On the Complexity of Multi-entry DFAs

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Table of Contents

Determinism & Nondeterminism

Determinism

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Initial state nondeterminism

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MDFAs

Definition (MDFA)

A Multi-entry Finite Automaton (MDFA) is a 5-tuple $M = \langle Q, \Sigma, \delta, I, F \rangle$ where

- Q is a finite set of states
- Σ is a finite alphabet
- $\delta : Q \times \Sigma \to Q$ is a deterministic transition function
- $I \subseteq Q$ is the subset of initial states
- $F \subseteq Q$ is the subset of final states

The language accepted by M is $\mathcal{L}(M) = \{w \in \Sigma^\star \mid \delta(I, w) \cap F \neq \emptyset\}$

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$$Q' = Q^3, \qquad I' = \{\langle q_0, q, q\rangle \mid q \in Q\}, \qquad F' = \{\langle q, f, q\rangle \mid f \in F\}$$
$$\delta'(\langle p, q, r\rangle, \sigma) = \langle \delta(p, \sigma), \delta(q, \sigma), r\rangle$$

A Brief History of MDFAs

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2004 - A. Malcher
Proved NP-completeness of minimisation by fixing $|I|$.

Table of Contents

Operational State Complexity

Table: Summary of operational state complexity results.

Operation	DFA	$\|\Sigma\|$	MDFA	$\|\Sigma\|$	NFA	$\|\Sigma\|$
$L \cap K$	nm	2	nm	2	nm	2
$L \cup K$	nm	2	**$n+m$**	2	$n+m$	2
$\Sigma^\star \setminus L$	n	1	2^n	4	2^n	2
$\mathrm{Suff}(L)$	$2^n - 1$	2	**n**	2	n	2
$\mathrm{Pref}(L)$	n	2	n	2	n	2
$\mathrm{Fact}(L)$	2^{n-1}	2	**n**	2	n	2
L^R	$2^n - 1$	2	$2^n - 1$	$\mathcal{O}(n^2)$	n	2

Table: ■ previous results ■ good ■ bad

Complementation – Upper Bound

Upper Bound: Follows from NFAs.

$$\begin{array}{c} M \\ n \text{ states} \end{array} \xrightarrow{\text{determinise}} \begin{array}{c} \mathcal{D}(M) \\ 2^n - 1 \text{ states} \end{array} \xrightarrow[\text{complement}]{\text{add sink state}} \begin{array}{c} \overline{\mathcal{D}(M)} \\ 2^n \text{ states} \end{array}$$

Complementation – Tightness

Tightness: is not so simple. . .

[1] "A Survey on Fooling Sets as Effective Tools for Lower Bounds on Nondeterministic Complexity" (2018)

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Lemma (M. Hospodár et al. [1])

Let N be an n-state NFA in which every subset of the state set is reachable and co-reachable. Then, every NFA for the complement of the language $\mathcal{L}(N)$ has at least 2^n states.

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We need a family of MDFAs M_n

- All subsets $S \subseteq Q$ are reachable
- All subsets $S \subseteq Q$ are co-reachable
- Minimal

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Brzozowski's Universal Witness

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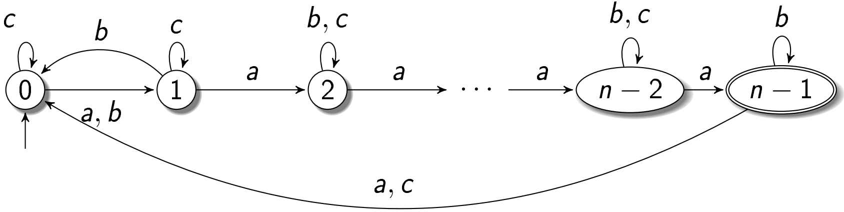

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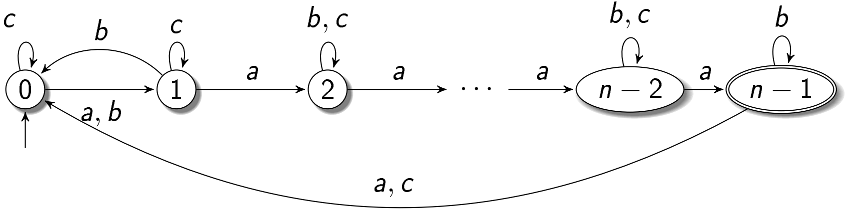
It turns out that (with simple modifications) it will also serve as a witness for MDFAs.



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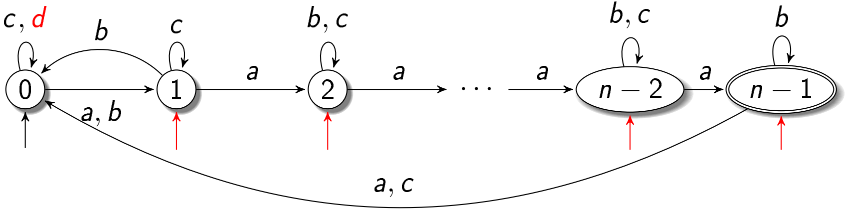

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Reachability

Lemma

If a subset $S \subseteq Q$ is reachable, then all other subsets of the same cardinality are also reachable.

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$$I \xrightarrow{w \in \Sigma^\star} S \xrightarrow{w_\pi \in \Sigma^\star} T$$

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Lemma

If a subset $S \subseteq Q$ with $|S| \geq 2$ is reachable, then all subsets of cardinality $|S| - 1$ are also reachable.

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$$\begin{tikzcd}
I \arrow[r, "w \in \Sigma^\star"] & S' \cup \{p, q\} \arrow[r, "w_\pi \in \Sigma^\star", "{\begin{array}{c} p \mapsto 0 \\ q \mapsto n-1 \end{array}}"'] & S' \cup \{0, n-1\} \arrow[r, "c \in \Sigma"] & S' \cup \{0\} \arrow[d, "w_{\pi'} \in \Sigma^\star"] \\
 & & & |T| = |S| - 1
\end{tikzcd}$$

Reachability

- $Q = I$ is reachable
- $\emptyset$ is reachable
- By iteration of the previous lemmata all subsets are reachable

Lemma

All subsets $S \subseteq Q$ are reachable.

Co-Reachability

A. Salomaa et al.[3] showed that all 2^n subsets are co-reachable if:

- $\mathcal{L}(M) \neq \emptyset$
- $\mathcal{L}(M) \neq \Sigma^\star$
- $|\Sigma| \geq 3$
- Its transition semigroup is functionally complete

Lemma

All subsets $S \subseteq Q$ are co-reachable.

[3] "On the state complexity of reversals of regular languages" (2004)

Computational Complexity

Table: Summary of the complexity of decision problems.

Problem	DFA	MDFA	NFA
Membership	L	**L**	NL-complete
Emptiness	NL-complete	NL-complete	NL-complete
Universality	NL-complete	PSpace-complete	PSpace-complete
Inclusion	P	PSpace-complete	PSpace-complete
Minimisation	P	PSpace-complete	PSpace-complete

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Universality is PSpace-complete

Membership: Follows from NFA-Universality

$$\begin{array}{c} \text{NFA-Universality} \\ \text{PSpace} \end{array} \xrightarrow{\text{MDFA}s \subseteq \textit{NFAs}} \begin{array}{c} \text{MDFA-Universality} \\ \text{PSpace} \end{array}$$

Universality is PSpace-complete

Completeness: Reduction from DFA-Intersection-Emptiness.

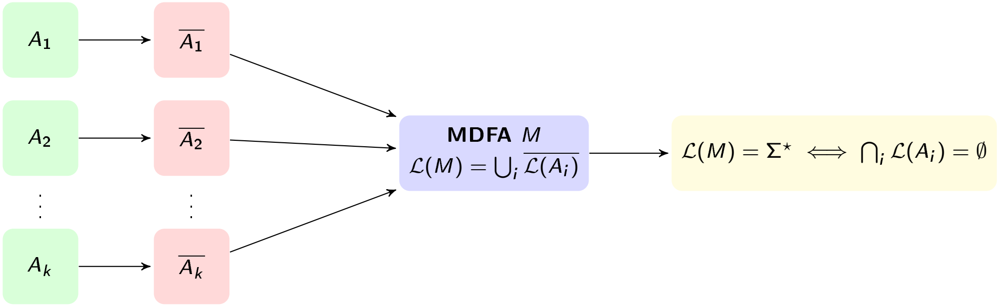

Table of Contents

Conclusion & Future Work

This research clarifies the role that nondeterminism plays in automata based models.

- How does limited nondeterminism behave under standard language-theoretic operations
- How does limited nondeterminism affect complexity-theoretic properties
- Fills a gap in the literature

Future Work

- Improve the alphabet size for complementation & reversal
- State complexity of (con)catenation and Kleene star
- State complexity of unary languages and other subregular families

Thank You!